

GROUP ACTIONS ON ASPHERICAL $A_k(N)$ -MANIFOLDS¹

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ABSTRACT. By an aspherical $A_k(N)$ -manifold, we mean a compact connected manifold M together with a map f from M into an aspherical complex N such that $f^*: H^k(N; Q) \rightarrow H^k(M; Q)$ is nontrivial. In this paper we shall show that if S^1 acts effectively and smoothly on a smooth aspherical $A_k(N)$ -manifold, $k > 1$, N a closed oriented Riemannian k -manifold, with strictly negative curvature, and the K -degree $K(f) \neq 0$, then the fixed point set F is not empty, and at least one component of $F = \bigcup_j F_j$ is an aspherical $A_k(N)$ -manifold. Moreover, $\text{Sign}(f) = \sum_j \text{Sign}(f|_{F_j})$. We also study the degree of symmetry and semisimple degree of symmetry of aspherical $A_k(N)$ -manifolds.

1. Introduction. Suppose M^m is a compact connected topological or differentiable m -manifold. Following [15], M is called an A_k -manifold, where k is a nonnegative integer, if there exists $w_i \in H^1(M, Q)$, $1 \leq i \leq k$, such that $w_1 \cup \cdots \cup w_k \neq 0$. Without loss of generality, in fact, we can assume that w_i 's belong to the free part of $H^1(M, Z)$. Let M be a compact connected differentiable manifold. The *degree of symmetry* $N(M)$ (resp. *semisimple degree of symmetry* $N^s(M)$) of M is defined as the supremum of the dimensions of all compact (resp. compact semisimple) Lie groups which can act smoothly and effectively on M . If M is a compact connected topological manifold, the *degree of symmetry* $N_T(M)$ and *semisimple degree of symmetry* $N_T^s(M)$ can be similarly defined by assuming the actions to be topological.

A space is called *aspherical* if its universal covering space is contractible. Burghlelea (cf. [21]) has proposed to compute or estimate $N(M)$ for a connected closed differentiable m -manifold M if there exists a degree one map $f: M^m \rightarrow N^m$, where N is a closed aspherical manifold. Considerable information has been obtained in relation to this problem. If $N = T^m$, the m -torus, then M is called *hypertoral* [20]. This was studied by Schultz [20, 21] and Gromov and Lawson [6]. One result of Conner and Raymond in [4] corresponds to the case $M = N$ and f is the identity map. If M is *hyperaspherical*, i.e., degree of f is nonzero, then Donnelly and Schultz [5] have shown that $N^s(M) = 0$. Schoen and Yau [19] have investigated the case when N is a closed Riemannian manifold of nonpositive curvature which is aspherical because its universal covering is diffeomorphic to a Euclidean space. In [5],

Received by the editors September 14, 1981 and, in revised form, September 2, 1982.

1980 *Mathematics Subject Classification*. Primary 57S10, 57S15; Secondary 57S25.

Key words and phrases. A_k -manifold, aspherical $A_k(N)$ -manifold, $\hat{A}$ -degree, K -degree, aspherical complex, Euler characteristic, degree of symmetry, semisimple degree of symmetry, complex manifold, Riemannian manifold, negative curvature, Pontrjagin number, Postnikov system, signature, Thom class.

¹Portions of this paper were presented in the Special Session on Topological Transformation Groups under the title " K -degree of a map and $A_k(N)$ -manifold" at the A.M.S. meeting in Amherst, Massachusetts, October 1981.

Donnelly and Schultz have proved the following topological modified version of the Schoen-Yau result:

1.1 THEOREM [5]. *Let M^m be a closed oriented m -manifold. Suppose there exists a map f from M^m to a closed oriented Riemannian manifold N of strictly negative curvature such that $f_*: H_k(M; Q) \rightarrow H_k(N; Q)$ is nontrivial for some $k > 0$. Then*

$$N_T(M) \leq \begin{cases} N_T(S^{m-k}) = \langle m - k + 1 \rangle_{SO} & \text{if } k > 1, \\ N_T(S^{m-1}) + 1 & \text{if } k = 1, \end{cases}$$

where $\langle s \rangle_{SO}$ denotes $\dim SO(s)$.

A theorem similar to the Schoen and Yau result [19] was also obtained by Browder and Hsiang [23]. Their paper also proved a “higher $\hat{A}$ -genus” theorem which is analogous to our Proposition 3.10.

Let M^m be a compact connected topological or differentiable m -manifold. We shall say that M is an $A_k(N)$ -manifold (resp. aspherical $A_k(N)$ -manifold) if N is a closed connected oriented manifold (resp. an aspherical complex), k a nonnegative integer, and there exists a continuous map $f: M \rightarrow N$ such that $f^*: H^k(N; Q) \rightarrow H^k(M; Q)$ is nontrivial. It follows from the definition that any connected closed manifold is an $A_0(N)$ -manifold (resp. aspherical $A_0(N)$ -manifold) for any manifold (resp. aspherical complex) N . The following results show that both the $A_k(N)$ -manifold and the aspherical $A_k(N)$ -manifold may be viewed as generalized A_k -manifolds.

1.2 THEOREM. *Let M be a compact connected manifold. Then M is an $A_k(T^k)$ -manifold if and only if M is an A_k -manifold.*

PROOF. Obviously, an $A_k(T^k)$ -manifold is an A_k -manifold. To prove sufficiency, let M be an A_k -manifold, and $w_i \in H^1(M; Z)$, $1 \leq i \leq k$, be such that $w_1 \cup \cdots \cup w_k \neq 0$. Since $H^1(M; Z) \cong [M; K(Z, 1)] = [M; S^1]$, for each w_i , there corresponds a map $f_i: M \rightarrow S^1 = S_i^1$ with $f_i^*\{S_i^1\} = w_i$; where $\{S_i^1\}$ denotes the fundamental cohomology class of S_i^1 . Set $f = \prod_{i=1}^k f_i: M \rightarrow \prod_{i=1}^k S_i^1 = T^k$. Then $f^*\{T^k\} = \prod_{i=1}^k w_i \neq 0$. That is, M is an $A_k(T^k)$ -manifold.

In this paper we shall investigate the transformation groups on aspherical $A_k(N)$ -manifolds. We shall introduce the notion of the Euler characteristic $\chi(f)$ and the K -degree $K(f)$ of a smooth map $f: M \rightarrow N$. If $\chi(f) \neq 0$ or $K(f) \neq 0$, then M is an $A_k(N)$ -manifold. Moreover, we shall show that if S^1 acts effectively and smoothly on a smooth $A_k(N)$ -manifold M , $k > 1$, with fixed point set F , and N a closed oriented Riemannian manifold with strictly negative curvature, then $\text{Sign}(f) = \sum_j \text{Sign}(f|_{F_j})$, where $F = \bigcup_j F_j$. Moreover, if $K(f) \neq 0$ for some K , then F is not empty and at least one component of F is also an aspherical $A_k(N)$ -manifold. We will also generalize some results in [15] from A_k -manifolds to aspherical $A_k(N)$ -manifolds. In particular, we shall obtain several generalizations of Theorem 1.1. For instance, if M is an aspherical $A_k(N)$ -manifold, $x \in H^\alpha(M; Q)$, and $y = f^*(\bar{y}) \in H^k(M; Q)$ such that $xy \neq 0$ and $\bar{m} = m - k \geq 19$, then

$$N_T(M) \leq k + \langle \bar{m} - \alpha + 1 \rangle_{SO} + \langle \alpha + 1 \rangle_{SO}$$

or

$$N_T(M) \leq k + \dim SU(\bar{m}/2 + 1).$$

If, in addition we assume that N is a closed oriented Riemannian manifold with strictly negative curvature, $k > 1$, and x , y and $\bar{m}$ are as above, then we have the following generalization of Theorem 1.1:

$$N_T(M) \leq \langle \bar{m} - \alpha + 1 \rangle_{SO} + \langle \alpha + 1 \rangle_{SO}$$

or

$$N_T(M) = \dim SU(\bar{m}/2 + 1), \quad M \approx CP^{\bar{m}/2} \times W^k.$$

These bounds on $N_T(M)$ are of course much sharper than $\langle \bar{m} + 1 \rangle_{SO}$, especially if α can be chosen near $[(\bar{m} + 1)/2]$. In §4, we shall define a numerical invariant $N(M; H)$ which is very useful to estimate the degree of symmetry of complex manifolds. We shall apply this invariant to show that if M is a complex aspherical $A_k(N)$ -manifold and $m = 2n + k$, then $N(M) \leq k + \langle n + 1 \rangle_{SU}$, where $\langle s \rangle_{SU} = \dim SU(s)$.

2. Existence of induced maps. The following result is a topological analogue of the fundamental theorem of homomorphisms for groups.

2.1 THEOREM. *Let M , N and W be CW complexes, $f: M \rightarrow N$ and $g: M \rightarrow W$ be continuous. Suppose g_{*i} is onto for $1 \leq i \leq \phi(N) = d$, where $g_{*i}: \pi_i(M) \rightarrow \pi_i(W)$ and $\phi(N) = \max\{j: \pi_j(N) \neq 0\} < \infty$. Then there exists a map $h: W \rightarrow N$ such that hg is homotopic to f if and only if $\text{Ker } g_{*i} \subset \text{Ker } f_{*i}$ for $1 \leq i \leq d$.*

PROOF. Suppose $\text{Ker } g_{*i} \subset \text{Ker } f_{*i}$ for $1 \leq i \leq d$. Let $\{a_n: M \rightarrow M_n\}$, $\{b_n: W \rightarrow W_n\}$ and $\{c_n: N \rightarrow N_n\}$ be the Postnikov systems of the complexes M , W and N , respectively. By definition we have homotopy commutative diagrams

$$\begin{array}{ccc} M & \xrightarrow{a_{n+1}} & M_{n+1} \\ & a_n \searrow & \downarrow \\ & & M_n \end{array}$$

and fibrations $K(\pi_{n+1}(M), n+1) \rightarrow M_{n+1} \rightarrow M_n$, etc. There exist maps $\{f_n: M_n \rightarrow N_n\}$ and $\{g_n: M_n \rightarrow W_n\}$ such that

$$\begin{array}{ccccc} M_{n+1} & \xrightarrow{f_{n+1}} & N_{n+1} & & M_{n+1} \xrightarrow{g_{n+1}} W_{n+1} \\ \downarrow & & \downarrow & \text{and} & \downarrow \\ M_n & \xrightarrow{f_n} & N_n & & M_n \xrightarrow{g_n} W_n \end{array}$$

commute, and

$$\begin{array}{ccccc} M & \xrightarrow{f} & N & & M \xrightarrow{g} W \\ a_n \downarrow & & \downarrow c_n & \text{and} & a_n \downarrow \\ M_n & \xrightarrow{f_n} & N_n & & M_n \xrightarrow{g_n} W_n \end{array}$$

are homotopy commutative. We shall inductively construct the maps $h_n: W_n \rightarrow N_n$ so that $h_n g_n \simeq f_n$. Assume that h_n has been constructed. Consider the following diagram where the vertical maps are fibrations:

$$\begin{array}{ccccc}
 K(\pi_{n+1}(M), n+1) & \xrightarrow{\tilde{f}_{n+1}} & K(\pi_{n+1}(N), n+1) & & \\
 \downarrow & \searrow \tilde{g}_{n+1} & \nearrow \tilde{h}_{n+1} & & \downarrow \\
 & & K(\pi_{n+1}(W), n+1) & & \\
 \downarrow & \xrightarrow{f_{n+1}} & \downarrow & \xrightarrow{\quad} & \downarrow \\
 M_{n+1} & \xrightarrow{g_{n+1}} & W_{n+1} & \xrightarrow{\tilde{h}_{n+1}} & N_{n+1} \\
 \downarrow & \searrow & \downarrow & \nearrow & \downarrow \\
 M_n & \xrightarrow{f_n} & W_n & \xrightarrow{h_n} & N_n
 \end{array}$$

By hypotheses and the fundamental theorem of homomorphism for groups, there exists a homomorphism $h_{*n+1}: \pi_{n+1}(W) \rightarrow \pi_{n+1}(N)$ such that $h_{*n+1} g_{*n+1} = f_{*n+1}$. Since

$$[K(\pi, n+1), K(\pi', n+1)] \cong \text{Hom}(\pi, \pi'),$$

there exists a map $\tilde{h}_{n+1}$ such that $\tilde{h}_{n+1} \tilde{g}_{n+1} \simeq \tilde{f}_{n+1}$. The maps $\tilde{h}_{n+1}$ and h_n induce a map $h_{n+1}: W_{n+1} \rightarrow N_{n+1}$ such that $f_{n+1} \simeq h_{n+1} g_{n+1}$. Since $d = \phi(N)$, c_d is a weak homotopy equivalence, hence it is a homotopy equivalence. Let $\phi: N_d \rightarrow N$ be a homotopy inverse of c_d . Then the map $h = \phi h_d b_d$ satisfies $hg \simeq f$.

If N is a $K(\pi_1(N), 1)$ -complex, i.e. aspherical complex, then $\phi(N) = 1$. The special case when N is an aspherical complex was proved in [5].

2.2 PROPOSITION. Assume that M is an aspherical $A_k(N)$ -manifold and $\pi_1(M)$ abelian. Let T^s act effectively on M with nonempty fixed point set $F(T^s, M)$. Then there exists a map $h: M/T^s \rightarrow N$ such that $f \simeq h\pi$ and $s \leq m - k$, where $\pi: M \rightarrow M/T^s$ is the natural projection.

PROOF. It is known that $\pi_{*1}: \pi_1(M) \rightarrow \pi_1(M/T^s)$ is surjective [2]. Thus $\pi_1(M/T^s)$ is abelian because $\pi_1(M)$ is abelian. Since $F(T^s, M)$ is not empty, $\pi^*: H^1(M/T^s; Q) \rightarrow H^1(M; Q)$ is surjective [2]. Equivalently, $\pi_*: H_1(M; Q) \rightarrow H_1(M/T^s; Q)$ is injective. But π_* is also surjective [2], hence π_* an isomorphism. Hence it is not difficult to see that $\text{Ker } \pi_{*1}$ is a finite group. Since $\pi_1(N)$ is torsion free, $\text{Ker } \pi_{*1} \subset \text{Ker } f_{*1}$. Since M/T^s has the homotopy type of a finite complex [5], it follows from Theorem 2.1 that there exists a map $h: M/T^s \rightarrow N$ such that $h\pi \simeq f$. As M is an aspherical $A_k(N)$ -manifold, $h^*: H^k(N; Q) \rightarrow H^k(M/T^s; Q)$ is nontrivial. Thus we have $k \leq \dim M/T^s = m - s$, or $s \leq m - k$ as desired.

2.3 PROPOSITION. Suppose that M is an aspherical $A_k(N)$ -manifold and G a compact semisimple Lie group acting almost effectively on M with $G(x)$ as a principal orbit. Then there exists a map $h: M/G \rightarrow N$ such that $f \simeq h\pi$ and $\dim G(x) \leq m - k$.

PROOF. Let $i: G \rightarrow M$ be the orbit map defined by $i(x) = x(m)$, $m \in M$. According to [5], we have an isomorphism

$$\pi_1(M/G) \cong (\pi_1(M)/i_{*1}\pi_1(G))/P,$$

where P is a finite normal subgroup of $\pi_1(M)/i_{*1}\pi_1(G)$. Since $\pi_1(N)$ is torsion free, we have $\text{Ker } \pi_{*1} \subset \text{Ker } f_{*1}$. Again $k \leq \dim M/G = m - \dim G(x)$, and the proof is complete.

From the proof of [5, Theorem 3.5] we can see that the following holds.

2.4 PROPOSITION. *Let M be an $A_k(N)$ -manifold, $k > 1$, and N a closed oriented Riemannian manifold of strictly negative curvature. Suppose G is a compact connected Lie group acting almost effectively on M with G/H as a principal orbit. Then there exists a map $h: M/G \rightarrow N$ such that $f \simeq h\pi$ and $\dim G/H \leq m - k$.*

3. Euler characteristic, K -degree of a map, and fixed point set. Let M^m and N^k be closed connected oriented manifolds, and $f: M \rightarrow N$ be a smooth map. Let $x \in N$ be a regular value of f , and $W = f^{-1}(x)$. Define the *Euler characteristic* $\chi(f)$ of the map f by $\chi(f) = \chi(W) \bmod 2$, where $\chi(W)$ denotes the Euler characteristic of W , and set $\chi(f) = 0$ if $W = \emptyset$. If $m = 4r + k$, and $K = \{K_n\}$ is a multiplicative sequence defined by Hirzebruch in [7], define the K -degree $K(f)$ of f to be the following number:

$$K(f) = \begin{cases} \langle K_r(W), [W] \rangle \in \mathcal{Q}, & \text{if } W \neq \emptyset, \\ 0 & \text{if } W = \emptyset. \end{cases}$$

Since the oriented cobordism class of W is independent of the choice of the regular value x , $K(f)$ is well defined. If $r = 0$, then $K_0 = 1$ and $K(f)$ is simply the degree of f . Define the *signature* of f by $\text{Sign}(f) = L(f)$, the L -genus of W . The special case $K = \hat{A}$ is defined in [6]. To prove that $\chi(f)$ is well defined, let $W' = f^{-1}(y)$, y is another regular value of f . Then there is a compact oriented manifold V with boundary $\partial V = W \cup W'$. But $H^i(V, W; \mathcal{Q}) \cong H_{v-i}^i(V, W'; \mathcal{Q})$ where $v = \dim V$. Hence, if v is odd, $\chi(V, W) = -\chi(V, W')$, and so $\chi(W) = \chi(W') \bmod 2$.

We shall denote the normal bundle of W in M by ν , $p: \nu \rightarrow M$ the projection and $\eta: M \rightarrow T(\nu)$ the natural collapsing map, where $T(\nu)$ is the Thom space of ν .

3.1 THEOREM. *Let $f: M^m \rightarrow N^k$ be a smooth map. Then*

- (a) $K(f) = \langle K_r(M) \cup f^*\{N\}, [M] \rangle$ if $m = 4r + k$,
- (b) $\chi(f) = \langle (p\eta)^*e(W) \cup f^*\{N\}, [M] \rangle \bmod 2$, where $[M]$ denotes the fundamental class of M , and $e(W)$ the Euler class of the tangent bundle TW . In particular, if $K(f) \neq 0$, or $\chi(f) \neq 0$, then M is an $A_k(N)$ -manifold.

PROOF. Let ν' denote the normal bundle of x in N and $U \in H^k(T(\nu))$ and $U' \in H^k(T(\nu'))$ be the Thom classes. The map f induces a bundle map $b: \nu \rightarrow \nu'$ and hence a map $T(b)$ such that $T(b)^*U' = U$ (cf. [3, II 2.8]). The natural collapsing map $\eta': N \rightarrow T(\nu')$ has degree 1, hence $\eta'^*U' = \{N\}$. Since $T(b)\eta = \eta'f$, it follows that

$$f^*\{N\} = f^*\eta'^*U' = \eta^*T(b)^*U' = \eta^*U.$$

By using the Poincaré duality we can easily show that $j_*[W] = [M] \cap \eta^*U$, where $j: W \rightarrow M$ is the inclusion. Since the normal bundle ν is trivial, hence $j^*K_r(M) = K_r(W)$. It follows that

$$\begin{aligned} K(f) &= \langle K_r(W), [W] \rangle = \langle j^*K_r(M), [W] \rangle = \langle K_r(M), j_*[W] \rangle \\ &= \langle K_r(M), [M] \cap \eta^*U \rangle = \langle K_r(M) \cup \eta^*U, [M] \rangle \\ &= \langle K_r(M) \cup f^*\{N\}, [M] \rangle. \end{aligned}$$

This completes the proof of (a).

Let $p: T(\nu) \rightarrow W$ be projection. Since η has degree 1, $\eta_*[M] \cap U = [W]$. It follows that

$$\langle e(W), [W] \rangle = \langle e(W), \eta_*[M] \cap U \rangle = \langle (p\eta)^*e(W) \cup f^*\{N\}, [M] \rangle.$$

Hence $\chi(f) = \langle (p\eta)^*e(W) \cup f^*\{N\}, [M] \rangle \bmod 2$.

The main result of this section is the following:

3.2 THEOREM. *Suppose M^m and N^k are closed oriented connected manifolds, where N is a Riemannian manifold with strictly negative curvature and $k > 1$. Let $f: M \rightarrow N$ be a smooth map such that $K(f) \neq 0$ for some multiplicative sequence K . Then for any smooth action of S^1 on M , the fixed point set F is not empty, and at least one component of F is an aspherical $A_k(N)$ -manifold. Moreover, we can orient each component F_j of F so that $\text{Sign}(f) = \sum_j \text{Sign}(f|F_j)$.*

From now on we shall always assume that N is an aspherical complex and call an aspherical $A_k(N)$ -manifold simply an $A_k(N)$ -manifold. In view of Theorem 3.1, Theorem 3.2 is a special case of the following:

3.3 THEOREM. *Let $G = S^1$ act effectively and smoothly on a smooth $A_k(N)$ -manifold, $k > 1$, and N a closed oriented Riemannian manifold with strictly negative curvature.*

(a) *Suppose $K(M)$ is a polynomial in the Pontrjagin classes of M with rational coefficients such that $\langle z \cup K(M), [M] \rangle \neq 0$ where $z = f^*(\bar{z}) \in H^k(M; \mathbb{Q})$. Then the fixed point set F of G is not empty, and at least one component of F is also an $A_k(N)$ -manifold.*

(b) *We can orient each component of $F = \bigcup_j F_j$ so that*

$$\text{Sign}(f) = \sum_j \text{Sign}(f|F_j).$$

This theorem is an immediate consequence of the following two theorems.

3.4 THEOREM. *Suppose $G = S^1$ acts effectively and smoothly on a smooth $A_k(N)$ -manifold M , and there exists a map $h: M/G \rightarrow N$ such that $f \simeq h\pi$.*

(a) *If $\langle z \cup K(M), [M] \rangle \neq 0$ where z and K are as in Theorem 3.3 and the fixed point set F of G is not empty. Then at least one component of F is also an $A_k(N)$ -manifold.*

(b) *We can orient each component F_j of F so that*

$$\text{Sign}(f) = \sum_j \text{Sign}(f_j), \quad f_j = f|F_j.$$